Group Theory
Week #5, Lecture #20

Group actions on sets (part IV)

$g \in G$, $x \in S$

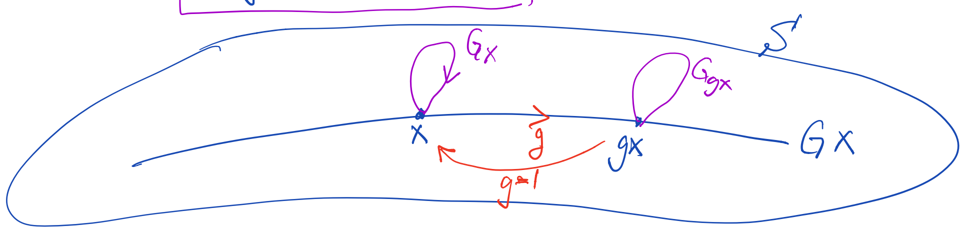
Setup:
- Group G acting on set S: $(g,x) \mapsto g * x = gx$
- $\mathrm{Orb}_G(x)$ • Orbit of x: $Gx := \{gx : g \in G\} \subseteq S$
- $\mathrm{Stab}_G(x)$ • Stabilizer subgroup of x: $G_x := \{g \in G : gx = x\} \leq G$
- $\mathrm{Fix}_G(S)$ • Fixed point set: $S^G := \{x \in S : gx = x, \forall g \in G\} \subseteq S$

Compatibility results regarding orbits & stabilizers

(1) The orbits of the group action partition S.

(2) The stabilizers along an orbit are conjugate to each other. More precisely:

$$\boxed{G_{gx} = g G_x g^{-1}}$$



reason: Suppose $h \in G_{gx}$, i.e., $h * (gx) = gx$

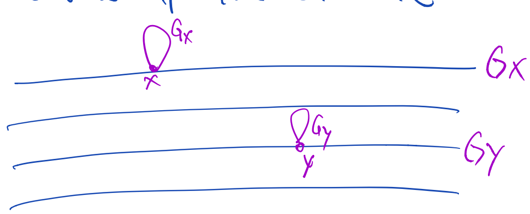
$\iff (hg)x = gx$

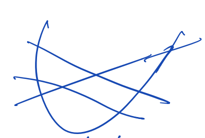
$\iff g^{-1}hg\, x = x$

$\iff g^{-1}hg \in G_x$

$\iff h \in gG_xg^{-1}$

(3) The orbits are in one-to-one correspondence with the left cosets of the stabilizers.





not the case!

Theorem (The Orbit-Stabilizer Theorem)

For a group G acting on a set S, we have a bijection, for every $x \in S$

$$\boxed{\left\{ \text{Orbit of } x \text{ under } G\text{-action} \right\} \longleftrightarrow \left\{ \text{left cosets of the stabilizer } G_x \right\}}$$

$$Gx \longleftrightarrow \{ gG_x \}$$

Proof $\boxed{\text{Fix } x \in S}$ Fix $g, h \in G$. Then:

$$gx = hx \iff (h^{-1}g)\,x = x \overset{\text{def of } G_x}{\iff} h^{-1}g \in G_x$$

$$\underset{\text{by def of left cosets}}{\iff} gG_x = hG_x$$

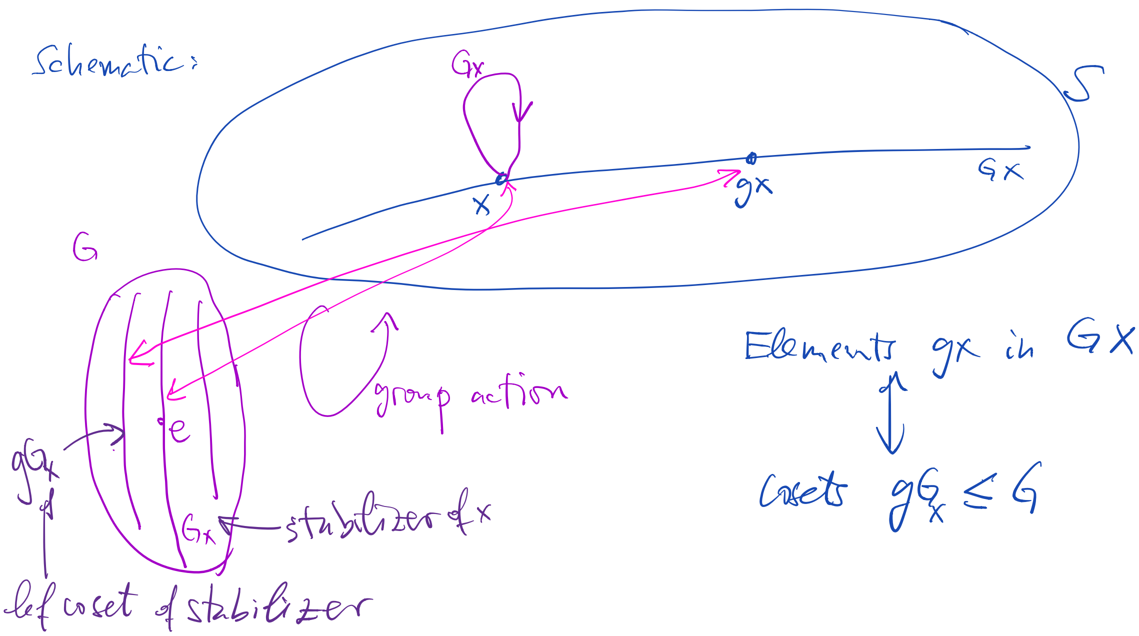
So we may define a function

$$\underset{\text{left coset of } G_x}{gG_x} \longmapsto \underset{\text{an element in the orbit } Gx}{gx}$$

with inverse $gx \longmapsto gG_x$.

By the above computation, both functions are well-defined ∎

Schematic:



Examples

(1) G acting on itself by conjugation

$$g * x = gxg^{-1}$$

Fix $x \in S = G$. Then

$$Gx = \{gxg^{-1} : g \in G\} = Cl(x) \quad \text{conjugacy class of } x$$

$$G_x = \{g \in G : gxg^{-1} = x\} = C(x) \quad \text{centralizer of } x$$

The Orbit-Stabilizer Theorem says:

$$\boxed{Cl(X) \xleftrightarrow[\text{bijection}]{} \{\text{left cosets of } C(x)\}}$$

$$\boxed{gxg^{-1} \longleftrightarrow g.C(x)}$$

Particular cases:

(i) $\boxed{G \text{ abelian}}$ $\quad Cl(x) = \{x\} \longleftrightarrow \{\text{left cosets of } C(x) = G\} = \{G\}$

$$x \longleftrightarrow G$$

(ii) $\boxed{G = S_3} = \langle a, b \mid a^3 = b^2 = 1,\ a^2b = ba\rangle = \{e, a, a^2, b, ab, a^2b\}$ Recall we have 3 conjugacy classes here, corresponding to the 3 distinct partitions of $n=3$ as a sum of integers ≥ 1:

partition	cycle shape	conjugacy class	
$1+1+1$	$(\,)$	$(\,)$	
$1+2$	(12)	$(12), (13), (23)$	
3	(123)	$(123), (132)$	

* Pick $x = (12) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = b$. Then

$$Gx = Cl(x) = \{(12), (13), (23)\} = \{b, ab, a^2b\}$$

$$G_x = C(x) = \{(\,), (12)\} = \{e, b\} = \langle b\rangle \cong \mathbb{Z}_2$$

$$\{\text{left cosets of } G_x\} = \{\langle b\rangle, a\langle b\rangle, a^2\langle b\rangle\}$$

$$= \{\{e, b\}, \{a, ab\}, \{a^2, a^2b\}\}$$

Orbit-stab correspondence:

$$b \longleftrightarrow \langle b\rangle$$
$$ab \longleftrightarrow a\langle b\rangle$$
$$a^2b \longleftrightarrow a^2\langle b\rangle$$

$$gbg^{-1} \longleftrightarrow g.\langle b\rangle$$

* Now pick $x=(123)=\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}=a$. Then

$$Gx = Cl(x) = \{(123),(132)\} = \{a,a^2\}$$

$$G_x = C(x) = \{(),(123),(132)\} = \{1,a,a^2\} = \langle a\rangle \cong \mathbb{Z}_3$$

$$\{\text{left cosets of } G_x\} = \{\langle a\rangle, b\langle a\rangle\} = \{\{e,a,a^2\},\{b,ba,ba^2\}\}$$

Orbit-stab corresp:

$$a \xleftrightarrow{g=1} \langle a\rangle \qquad gag^{-1} \longleftrightarrow g\cdot\langle a\rangle$$

$$bab^{-1} \checkmark = a^2 \xleftrightarrow{g=b} b\langle a\rangle$$

Theorem (Class Equation)

Let G be a finite group acting on a (finite) set S.

Then:

$$|S| \underset{(1)}{=} \sum_{\text{distinct orbits}} |Gx|$$

$$\underset{(2)}{=} |S^G| + \sum_{|Gx|>1} [G:G_x]$$

Proof (1) follows at once from $S = \coprod_{\text{distinct orbits}} Gx$

(2) Orbits may have size 1 or greater than 1

- Those of size 1 comprise the fixed point set.
- The other ones, by the Orbit-Stab. Theorem, are of size

$$|Gx| = |\{\text{left cosets of } G_x\}| \qquad \text{(OS Thm)}$$

$$= [G:G_x] = \frac{|G|}{|G_x|} \qquad \text{(Lagrange Thm)}$$

QED

Example $G = GL_2(\mathbb{Z}_p)$ acting on $S = \mathbb{Z}_p^2$ by left multiplication (p a prime)

$$A * V = AV$$

for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a,b,c,d \in \mathbb{Z}_p$, $ad - bc \neq 0$

$V = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad x, y \in \mathbb{Z}_p$

Two types of orbits / stabilizers:

* $V = 0 \longrightarrow A \cdot 0 = 0 \longrightarrow \begin{cases} GV = \{0\} \\ G_V = G \end{cases}$

* $V \neq 0 \longrightarrow \forall W \overset{\neq 0}{\in} \mathbb{Z}_p^2$, $\exists A \in G$ s.t. $AV = W$
(by solving a system of linear eqs)
$\longrightarrow \begin{cases} GV = \mathbb{Z}_p^2 \setminus \{0\} \\ |G_V| = p^2 - p \end{cases}$ by (*), or
$|G_V| = \frac{|G|}{|GV|} = \frac{(p^2-1)(p^2-p)}{p^2-1} = p^2 - p$

Class Equation:

$$|S| = |S^G| + \sum_{|Gx|>1} [G : G_x]$$

$$|\mathbb{Z}_p^2| = |\{0\}| + \sum_{\text{single orbit} = \mathbb{Z}_p^2 - \{(0)\}} [G : G_V] \qquad |G| = (p^2-1)(p^2-p)$$

$$p^2 = 1 + (p^2 - 1)$$

$$p^2 = p^2 \quad \checkmark$$

(*) Take $V = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. To find $G_V = \{A \mid AV = V\}$, we need to solve $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \begin{cases} a = 1 \\ c = 0 \end{cases}$

$\therefore A = \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix}$ $\quad d \in \mathbb{Z}_p^*$, $b \in \mathbb{Z}_p$ $\Longrightarrow |G_V| = p(p-1)$